AFCRC-TN-56-561

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-91

Reflectionless Transmission through Dielectrics and Scattering Potentials

I. KAY and H. E. MOSES

CONTRACT NO. AF 19(122)-42
MAY, 1956

c.2

NEW YORK UNIVERSITY

Institute of Mathematical Sciences

Division of Electromagnetic Research

Research Report No. EM-91

REFLECTIONLESS TRANSMISSION THROUGH DIELECTRICS AND SCATTERING POTENTIALS

I. Kay and H.E. Moses

Irwin Kay
I. Kay

H. E. Moses
H.E. Moses

Morris Kline
Morris Kline
Project Director

May, 1956

The research reported in this document has been made possible through support and sponsorship extended by the Air Force Cambridge Research Center, under Contract No. AF 19(122)-42. It is published for technical information only and does not necessarily represent recommendations or conclusions of the sponsoring agency.

New York, 1956

Abstract

A complete solution is given for the problem of constructing a plane stratified dielectric medium having the property that at a fixed frequency and polarization a plane wave at any angle of incidence will be transmitted without reflection by the medium. This problem can also be interpreted quantum mechanically. The dielectric medium of the first interpretation becomes, in the second interpretation, a potential for the one-dimensional Schrödinger equation such that a particle which is sent in from $-\infty$ is transmitted with probability one of getting through to $+\infty$, no matter what the initial kinetic energy of the particle may be.

In addition, the results can be applied to the problem of finding perfectly matched non-uniform transmission lines.

Table of Contents

		Page
1.	Introduction	1
2.	Construction of V(x)	4
3.	Properties of V(x)	8
4.	Example	10
5.	Conclusion and remarks	11
	Appendices	13
	References	16

1. Introduction

A difficult problem of practical as well as theoretical interest is the construction of a solid dielectric medium which is perfectly transparent to electromagnetic radiation. Generally the index of refraction of the medium is required to vary between set minimum and maximum values, and the medium itself must be confined to a slab of finite thickness bounded on either side by a vacuum.

The problem as stated, however, is an ideal one which seems impossible to solve. The most one can hope to do is to guarantee that a large percentage of the incident energy in a given frequency band over a given range of incidence angles will be transmitted with a correspondingly small percentage of the energy reflected by the medium.

As a start toward investigating this restricted, but still difficult, version of the problem it seems reasonable to look for dielectric media which are transparent in an even more limited sense. Thus, we shall consider an infinite class of dielectric media having the following properties:

(1) Each medium in the class will have a plane stratified variable index of refraction extending from $-\infty$ to $+\infty$;

(2) At all angles of incidence but at a fixed frequency and polarization a plane wave incident from $-\infty$ will be transmitted to $+\infty$ without reflection;

(3) The index of refraction of each medium will vary continuously throughout and will approach unity at both $-\infty$ and $+\infty$.

There is some hope that the results to be given here can be applied to the more complicated practical problem after further investigation. That is, in the infinite class of dielectrics we shall obtain there may be one or more which satisfies all requirements of the original problem well enough to be of use.

We start with the wave equation in a plane stratified medium with an index of refraction $n(x)$, $-\infty < x < \infty$:

$$\triangle w(x,y,z,k) + k^2 n^2(x)\, w(x,y,z,k) = 0 . \tag{1.1}$$

It is convenient for physical reasons to assume that the index of refraction has the form

$$n^2(x) = 1 + \mu^2(x) . \tag{1.2}$$

For an incident plane wave we can assume that the solution of (1.1) has the form

$$w(x,y,z,k) = u(x,k) \exp\left[ik(y \cos \beta + z \cos \gamma)\right] , \tag{1.3}$$

if the propagation direction is given by the direction cosines: $\cos \alpha$, $\cos \beta$, $\cos \gamma$. Using (1.2) and (1.3) in (1.1) we arrive at

$$\frac{d^2 u(x,E)}{dx^2} + \left[E - V(x)\right] u(x,E) = 0 , \tag{1.4}$$

where

$$E = k^2 \cos^2 \alpha \quad \text{and} \quad V(x) = -k^2 \mu^2(x) . \tag{1.5}$$

We note that $V(x)$ must be negative or zero for all x for any physically realizable medium.

Our problem now is to find a function $V(x)$ such that for a solution $u(x,E)$ of (1.4) having the property that

$$u(x,E) \sim \exp\left[i \sqrt{E}\, x\right] \qquad \text{as } x \to -\infty$$

we shall have

$$u(x,E) \sim t(E) \exp\left[i \sqrt{E}\, x\right] \qquad \text{as } x \to +\infty . \tag{1.6}$$

We shall see in what follows that an infinite class of such functions V(x) can be constructed. In addition, it will be proved that each V(x) is continuous and everywhere negative for all real x and that V(x) approaches zero exponentially as x approaches $+\infty$ or $-\infty$. Every V(x) of the infinite class of functions, as well as the corresponding solution u(x,E) of (1.4), will turn out to be an elementary function of x. It will be shown in Appendix 1 that these functions V(x) are the only possible ones satisfying our conditions.

Note that (1.4) is the one-dimensional Schrödinger equation. Another physical interpretation of the functions V(x) we shall construct is therefore possible: each such V(x) is a potential which has the property that it will allow an incident particle to pass through it with probability one, no matter what the particle's initial kinetic energy may be.*

Another possible application of the results might also be mentioned at this point. There exists a transformation [2] which changes an equation of the form (1.4) into the transmission-line equations for a two-wire line with variable capacitance and inductance. In this case the transmission line corresponding to any V(x) we obtain having properties just described will be matched perfectly at all frequencies, so that a voltage or current wave sent in from the left will travel toward the right with no reflection. The necessary transformation will be given in Appendix 2.

---

* The situation here resembles one described by Bargmann [1] for the radial Schrödinger equation with zero angular momentum. Bargmann has produced an example of a potential for the radial equation such that the corresponding zero angular momentum asymptotic phase shift is zero. In spite of their superficial resemblance, however, the problem considered here and the example of Bargmann are quite different in several important respects.

2. Construction of V(x)

To obtain a 2N-parameter family of functions V(x) each of which has the desired properties we first choose N arbitrary positive constants A_n and N arbitrary positive constants κ_n. We shall assume that a corresponding solution u(x,E) of (1.4) has the form

$$u(x,E) = \left\{1 + \sum_{n=1}^{N} \left[f_n(x)/(\kappa_n + i\sqrt{E}\,)\right] \exp\left[\kappa_n x\right]\right\} \exp\left[i\sqrt{E}\,x\right] . \tag{2.1}$$

The functions $f_n(x)$ are defined to be solutions of the N linear algebraic equations

$$\left\{A_n \exp\left[\kappa_n x\right] \sum_{\nu=1}^{N} \left[\exp\left[\kappa_\nu x\right]/(\kappa_n + \kappa_\nu)\right] f_\nu(x)\right\} + f_n(x) + A_n \exp\left[\kappa_n x\right] = 0 , \tag{2.2}$$

$$n = 1, \ldots, N .$$

It will turn out that the differential equation (1.4) has N discrete eigenvalues, $E = -\kappa_n^2$, and the functions $f_n(x)$ are the corresponding eigenfunctions.

Before we can prove the main result it is necessary to show that the system (2.2) has a unique solution for the set of N functions $f_n(x)$:

Lemma 1. The system (2.2) has a non-singular coefficient matrix whose determinant is positive for all finite x.

Proof. The coefficient matrix of (2.2) can be written I + A, where

$$\begin{aligned} I &= \|\delta_{n\nu}\| , \text{ the unit matrix,} \\ A &= \left\|A_n \exp\left[\kappa_n + \kappa_\nu\right]/(\kappa_n + \kappa_\nu)\right\| . \end{aligned} \tag{2.3}$$

We apply the diagonal transformation

$$D = \left\|A_n^{-1/2}\, \delta_{n\nu}\right\| \tag{2.4}$$

to (2.2), obtaining the result

$$(2.5)\qquad \sum_{\nu=1}^{N} \left\{ A_n^{1/2} A_\nu^{1/2} \exp\left[(\varkappa_n + \varkappa_\nu)x\right] \Big/ (\varkappa_n + \varkappa_\nu) \right\} g_\nu(x) + g_n(x) + A_n^{1/2} \exp\left[\varkappa_n x\right] = 0 ,$$

where we define $g_n(x) = A_n^{-1/2} f_n(x)$. For the symmetric matrix of (2.5) we write

$$(2.6)\qquad \hat{A} = \left\| A_n^{1/2} A_\nu^{1/2} \exp\left[(\varkappa_n + \varkappa_\nu)x\right] \Big/ (\varkappa_n + \varkappa_\nu) \right\| = D A D^{-1} .$$

Since, according to (2.6), the matrices A and $\hat{A}$ are similar, we have

$$(2.7)\qquad \det A = \det \hat{A}$$

and also

$$\det (A + I) = \det (\hat{A} + I) .$$

We can now work with the symmetric coefficient matrix $\hat{A} + I$ of (2.5).

To prove that $\det (\hat{A} + I) > 0$, it is sufficient to prove that $\det \hat{A} > 0$. We shall show, in fact, that $\hat{A}$ is a positive definite matrix; that is, given any set of N numbers y_n,

$$(2.8)\qquad \sum_{n,\nu=1}^{N} y_n \bar{y}_\nu A_n^{1/2} A_\nu^{1/2} \exp\left[(\varkappa_n + \varkappa_\nu)x\right] \Big/ (\varkappa_n + \varkappa_\nu) > 0 ,$$

unless all the y_n are zero.

The left side of (2.8) can be written

$$(2.9)\qquad \int_{-\infty}^{x} \sum_{n,\nu=1}^{N} A_n^{1/2} A_\nu^{1/2} y_n \bar{y}_\nu \exp\left[(\varkappa_n + \varkappa_\nu)z\right] dz = \int_{-\infty}^{x} \left| \sum_{n=1}^{N} A_n^{1/2} y_n \exp\left[\varkappa_n z\right] \right|^2 dz .$$

Expression (2.9) is always positive unless

$$\sum_{n=1}^{N} A_n^{1/2} y_n \exp\left[\varkappa_n z\right] = 0$$

for all z between $-\infty$ and x. It follows at once that (2.9) is positive unless

every y_n is zero, and thus Lemma 1 is proved.

The main result can now be proved:

Theorem 1. The function $u(x,E)$ defined in (2.1) and (2.2) satisfies the differential equation

$$d^2u/dx^2 + \left[E - V(x)\right]u = 0 , \tag{2.10}$$

where

$$V(x) = -2\, d^2\left\{\log \det (\hat{A} + I)\right\} \Big/ dx^2 \tag{2.11}$$

and $\hat{A}$ is the symmetric matrix of (2.6).

Proof. For convenience we define two symbols L and M_n to represent the differential operators

$$L = d^2/dx^2 + E - V(x)$$

and

$$M_n = d^2/dx^2 - \left[\varkappa_n^2 + V(x)\right] .$$

We substitute the expression for $u(x,E)$ given by (2.1) into the expression to the left of (2.10). We obtain

$$Lu = \left\{\sum_{n=1}^{N} \left[M_n f_n(x)\right] \exp\left[\varkappa_n x\right] \Big/ (\varkappa_n + i\sqrt{E}) + 2d\left[\sum_{n=1}^{N} f_n(x) \exp\left[\varkappa_n x\right]\right] \Big/ dx - V(x)\right\} \exp\left[i\sqrt{E}\, x\right] \tag{2.12}$$

Obviously (2.12) is zero if $f_n(x)$ satisfies

$$M_n f_n(x) = 0 \tag{2.13}$$

and also

$$V(x) = 2d\left\{\sum_{n=1}^{N} f_n(x) \exp\left[\varkappa_n x\right]\right\} \Big/ dx . \tag{2.14}$$

We apply the operator M_n to the n-th equation of (2.2) and substitute the expression

for $V(x)$ given by (2.14). The result is

$$(2.15)\qquad M_n f_n(x) + A_n \exp\left[\varkappa_n x\right] \cdot \sum_{\nu=1}^{N} \left[M_\nu f_\nu(x)/(\varkappa_n + \varkappa_\nu)\right] \exp\left[\varkappa_\nu x\right] = 0 .$$

Equations (2.15) are a homogeneous system of linear algebraic equations for the quantity

$$M_n f_n(x)$$

with coefficient matrix $A + I$. By Lemma 1, this matrix has an inverse and therefore

$$M_n f_n(x) = 0 .$$

Thus, (2.13) is satisfied by the $f_n(x)$ as defined in (2.2) when $V(x)$ has the definition (2.14).

At this stage we know that for $V(x)$ given by (2.14) the $u(x,E)$ defined by (2.1) and (2.2) satisfies (2.10). It remains to show that (2.14) and (2.11) are equivalent statements. To do this we write the solutions $f_n(x)$ of (2.2) in determinant form and put them together in the expression

$$\sum_{n=1}^{N} f_n(x) \exp\left[\varkappa_n x\right] .$$

By inspection it can be seen that

$$(2.16)\qquad \sum_{n=1}^{N} f_n(x)\exp\left[\varkappa_n x\right] = -d\left\{\log \det(A+I)\right\}/dx = -d\left\{\log \det(\hat{A}+I)\right\}/dx .$$

The theorem follows at once from (2.14).

3. Properties of V(x)

It follows from Lemma 1 and (2.11) that V(x) is bounded and continuous for all x.

Now we shall demonstrate two important properties of the function V(x) constructed in Theorem 1. To do so we shall need another lemma.

Lemma 2. $\det\left\{\hat{A}+I\right\}$ has the form

$$\det\left\{\hat{A}+I\right\} = 1 + \sum_n \alpha_n \exp\left[\beta_n x\right] , \tag{3.1}$$

where each α_n and β_n is a positive constant.

Proof. We recall that

$$\hat{A} = \left\| A_n^{1/2} A_\nu^{1/2} \exp\left[(\varkappa_n+\varkappa_\nu)x\right]/(\varkappa_n+\varkappa_\nu)\right\| , \tag{3.2}$$

and thus we have

$$\det \hat{A} = \left(\det\left\| (\varkappa_n+\varkappa_\nu)^{-1}\right\|\right)\left(\prod_{n=1}^{N} A_n\right) \exp\left[2\left(\sum_{n=1}^{N} \varkappa_n\right)x\right] . \tag{3.3}$$

Also, since $\| (\varkappa_n+\varkappa_\nu)^{-1} \|$ has the form of $\hat{A}$, by Lemma 1 it has a positive determinant. Therefore, by (3.3), $\det \hat{A}$ has the form $\alpha \exp[\beta x]$ with α and β positive constants. The same result holds for every principle minor of $\hat{A}$ since the principle minors all have the same form as $\det \hat{A}$.

Now the characteristic polynomial of $\hat{A}$ is

$$\det\left\{\hat{A}+\lambda I\right\} = \lambda^N + a_1\lambda^{N-1} + \ldots + a_N , \tag{3.4}$$

where a_r is the sum of the principle minors of $\hat{A}$ of order r (e.g., $a_N = \det \hat{A}$ and $a_1 = \operatorname{tr} \hat{A}$). If we set $\lambda = 1$, the lemma follows.

One would expect, physically, that if V(x) were ever positive, there would have to be a non-zero reflection for values of E below the positive maximum

of V(x). This is contrary to the fact, obvious from Theorem 1, that u(x,E) contains no reflected part. Actually we can make a slightly stronger statement.

Theorem 2. V(x) is negative for all finite x.

Proof. Writing

$$\Delta = \det \hat{A} + I \quad ,$$

we have

$$V(x) = -2\left(\Delta\Delta'' - \Delta'^2\right)/\Delta^2 \tag{3.5}$$

from (2.11). The primes mean differentiation with respect to x. We must show, then, that

$$\Delta\Delta'' - \Delta'^2 > 0$$

for all finite x. We have by Lemma 2,

$$\begin{aligned}\Delta\Delta'' - \Delta'^2 &= \left[1 + \sum_n \alpha_n \exp\left[\beta_n x\right]\right]\left[\sum_n \beta_n^2 \alpha_n \exp\left[\beta_n x\right]\right] - \left[\sum_n \beta_n \alpha_n \exp\left[\beta_n x\right]\right]^2 \\ &= \sum_n \beta^2 \alpha_n \exp\left[\beta_n x\right] + \frac{1}{2}\sum_{n,\nu} \alpha_n \alpha_\nu (\beta_n - \beta_\nu)^2 \exp\left[(\beta_n + \beta_\nu)x\right] \quad .\end{aligned} \tag{3.6}$$

By Lemma 2 the α_n and β_n are positive; therefore, (3.6) is positive. This completes the proof of Theorem 2.

Finally, let us examine the asymptotic behavior of V(x). We state

Theorem 3. V(x) approaches zero exponentially as x approaches either $+\infty$ or $-\infty$.

Proof. From the proof of Theorem 2 we have for the form of V(x):

$$V(x) = -2\left\{\sum_n \beta_n^2 \alpha_n \exp\left[\beta_n x\right] + \frac{1}{2}\sum_{n,\nu} \alpha_n\alpha_\nu(\beta_n - \beta_\nu)^2 \exp\left[(\beta_n + \beta_\nu)x\right]\right\}$$

(3.7)

$$\cdot\left\{1 + 2\sum_n \alpha_n \exp\left[\beta_n x\right] + \sum_{n,\nu} \alpha_n\alpha_\nu \exp\left[(\beta_n + \beta_\nu)x\right]\right\}^{-1} .$$

Now suppose β_μ is the largest of the constants β_n. Then the term in the denominator of (3.7) of largest exponential growth as x approaches $+\infty$ is that for which $n = \nu = \mu$, that is, for which the exponent is $2\beta_\mu$. This term is obviously missing from the numerator of (3.7); thus, the denominator grows faster than the numerator, and V(x) approaches zero as x approaches $+\infty$. As x approaches $-\infty$, V(x) obviously approaches zero.

4. Example

Consider the case of one constant A_1 and one constant $\varkappa_1$. The sytem (2.2) will consist of a single equation:

$$A_1 \exp\left[\varkappa_1 x\right] + f_1(x) + \left[A_1 \exp\left[2\varkappa_1 x\right]/2\varkappa_1\right] f_1(x) = 0 . \tag{4.1}$$

Thus

$$f_1(x) = -A_1 \exp\left[\varkappa_1 x\right] \Big/ \left[1 + A_1 \exp\left[2\varkappa_1 x\right]/2\varkappa_1\right] , \tag{4.2}$$

$$u(x,E) = \left\{1 - A_1 \exp\left[2\varkappa_1 x\right] \Big/ \left[1 + A_1 \exp\left[2\varkappa_1 x\right]/2\varkappa_1\right]\left[\varkappa_1 + i\sqrt{E}\right]\right\} \exp\left[i\sqrt{E}\, x\right] , \tag{4.3}$$

and from (2.11), since $\det\left\{I + \hat{A}\right\} = 1 + A_1 \exp\left[2\varkappa_1 x\right]/2\varkappa_1$, we have

$$V(x) = -4\varkappa_1 A_1 \exp\left[2\varkappa_1 x\right] \Big/ \left[1 + A_1 \exp\left[2\varkappa_1 x\right]/2\varkappa_1\right]^2 . \tag{4.4}$$

We observe that $V(x)$ has a single finite minimum and no finite maximum. It is interesting to find the position and value of the minimum of $V(x)$ in terms of the two parameters A_1 and $\varkappa_1$. We have for $dV/ds\big|_{x=x_o} = 0$:

$$-8\varkappa_1^2 A_1 \exp\left[2\varkappa_1 x_o\right]\left[1 + A_1 \exp\left[2\varkappa_1 x_o\right]/2\varkappa_1\right] + 8\varkappa_1 A_1^2 \exp\left[4\varkappa_1 x_o\right] = 0$$

for the position x_o of the minimum. Solving for x_o we have

$$x_o = \left[\log(2\varkappa_1/A_1)\right]/2\varkappa_1 \quad . \tag{4.5}$$

At its minimum position,

$$V(x_o) = -2\varkappa_1^2 \quad . \tag{4.6}$$

The value of the minimum depends only on $\varkappa_1$. Thus, for a prescribed minimum value of $V(x)$ the position of this minimum can be chosen arbitrarily by a proper choice of A_1.

5. Conclusion and remarks

Formulas (2.6) and (2.11) enable us to determine a 2N-parameter family of functions $V(x)$ which in turn determines a 2N-parameter family of dielectrics through (1.2). Formula (2.1), along with (1.3) and (1.4), demonstrates that the dielectric media are all transparent in the restricted sense described in the Introduction.

Since $V(x)$ is everywhere negative there must be at least one discrete eigenvalue in the spectrum of (1.4). We can actually say more than this: since in the proof of Theorem 1 we saw that $M_n f_n(x) = 0$, where M_n was defined to be the operator

$$d^2/dx^2 + E - V(x)$$

with E replaced by $-\kappa_n^2$, and since each $f_n(x)$ is obviously a square integrable function in the interval $-\infty < x < +\infty$, it follows that there must be N discrete eigenvalues

$$E = -\kappa_n^2 , \qquad n = 1, \ldots, N$$

with corresponding eigenfunctions $f_n(x)$.

The example of Section 4 had a single eigenvalue. Now we can make the minimum of V(x) in that example as large as we please in absolute value, according to (4.6), and the larger we make it the narrower will be the interval on which V(x) differs appreciably from zero. This is true because the area between V(x) and the x-axis is necessarily limited by the presence of only one eigenvalue. To get a 'broader' V(x) we would have to consider a case with more parameters A_n and κ_n.

From (4.3) we see that

$$u(x,E) \sim \exp\left[i \sqrt{E}\, x\right] \qquad \text{as } x \to -\infty$$

(5.1)

$$u(x,E) \sim \left[(i \sqrt{E} - \kappa_1)/(i \sqrt{E} + \kappa_1)\right] \exp\left[i \sqrt{E}\, x\right] \qquad \text{as } x \to +\infty .$$

The transmission coeffieicnt (the coefficient of the exponent) has the absolute value unity, as it should. Moreover, it has a pole at $E = -\kappa_n^2$ (the eigenvalue), as standard theory tells us it should. It can be seen easily that in general the transmission coefficient is always independent of the constants A_n; i.e., it is a function of the κ_n alone.

Acknowledgment

The authors wish to express appreciation to Prof. J.B. Keller for his discussions concerning this work and for his very helpful suggestions.

Appendix 1: Proof that the V(x) we have constructed are the only ones which there is no reflection

It has been shown [3] that a necessary condition for a suitable function V(x) in (1.4) is

$$V(x) = 2dK(x,x)/dx \tag{A.1}$$

where K(x,y) is a function of x and y which satisfies an integral equation

$$0 = R(x+y) + K(x,y) + \int_{-\infty}^{x} R(y+z)\, K(x,z)dz \tag{A.2}$$

for all real values of x and for values of y between $-\infty$ and x. The function R(x+y) is defined by

$$R(x+y) = (2\pi)^{-1} \int_{-\infty}^{\infty} r(k)\exp\left[-ik(x+y)\right]dk - i\sum_{n=1}^{N} r_n \exp\left[-ik_n(x+y)\right], \tag{A.3}$$

where r(k) is the complex reflection coefficient of (1.4), $E = k_n^2 = -\kappa_n^2$ are the discrete eigenvalues of (1.4), and the r_n are certain numbers on the positive imaginary axis. Since in our problem $r(k) \equiv 0$, (A.2) becomes on writing $r_n = iA_n$

$$0 = \sum_{n=1}^{N} A_n \exp\left[\kappa_n(x+y)\right] + K(x,y) + \sum_{n=1}^{N} A_n \int_{-\infty}^{x} K(x,z)\exp\left[\kappa_n(y+z)\right]dz\,. \tag{A.4}$$

It is obvious from (A.4) that K(x,y) has the form

$$K(x,y) = \sum_{n=1}^{N} f_n(x) \exp\left[\kappa_n y\right]\,. \tag{A.5}$$

Substitution of (A.5) into (A.4) produces the system of equations (2.2) for the functions $f_n(x)$. By Lemma 1 the system has a unique solution; therefore, K(x,y) and hence V(x) are unique for a given R(x+y) of the form assumed. When $r(k) \equiv 0$ this form is the only one possible for R(x+y).

Appendix 2

Consider the inhomogeneous transmission-line equations with variable capacitance $C(z)$ and variable inductance $L(z)$:

$$\begin{aligned} dV(z)/dz &= i\omega L(z)\, I(z) \\ dI(z)/dz &= i\omega C(z)\, V(z) , \end{aligned} \tag{A.6}$$

where $V(z)$ is the voltage, $I(z)$ the current, and ω the circular frequency. We can define a local characteristic impedance

$$Z(z) = \left[L(z)/C(z)\right]^{1/2} \tag{A.7}$$

and a local wave velocity

$$v(z) = \left[L(z)C(z)\right]^{-1/2} . \tag{A.8}$$

Let

$$\Psi(z) = \left[L(z)\right]^{-1/2} . \tag{A.9}$$

Then with the changes of variables

$$\begin{aligned} V(z) &= \left[Z(z)\right]^{1/2} U(z) , \\ x &= \int_{-\infty}^{z} \left[1/v(z)\right] dz \end{aligned} \tag{A.10}$$

we have

$$d^2U/dx^2 + \left[\omega^2 - P(x)\right]U = 0 , \tag{A.11}$$

where $P(x)$ is related to $\Psi(z(x))$ by

$$d^2\Psi/dx^2 - P(x)\Psi = 0 . \tag{A.12}$$

Note that (A.11) has exactly the same form as (1.4) with $P(x)$ replacing

the 'potential' V(x) there. Moreover, (A.12) is the same as (A.11) with $\omega = 0$ and U replaced by Ψ.

To use this result we observe that if $x \to -\infty$ when $z \to -\infty$ and if $x \to +\infty$ when $z \to +\infty$, x being a monotonic function of z (these conditions must be satisfied), then when P(x) represents a reflectionless potential the transmission line given by (A.6) is reflectionless. To get the characteristic impedance we choose the solution Ψ of (A.12) such that $\Psi \to Z_o^{-1/2}$ as $x \to -\infty$, where Z_o is any desired limiting value of the characteristic impedance at $-\infty$, and $d\Psi/dx$ approaches zero as $x \to -\infty$. We have shown how to construct a solution of (A.11) as well as the P(x); thus, we can obtain Ψ by considering the special case of (A.11) for which $\omega = 0$.

Note that our procedure determines only the characteristic impedance Z. The wave velocity v can be chosen more or less arbitrarily provided it is such that x and z are monotonically related as assumed.

References

[1] Bargman, V. - On the connection between phase shifts and scattering potentials; Rev. Mod. Phys., 21, 488 (1949).

[2] Kay, I. - The inverse scattering problem; New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. EM-74, 1955, pp. 26-38.

[3] Kay, I. and Moses, H.E. - The determination of the scattering potential from the spectral measure function, III; Il Nuovo Cimento, Ser. X, 3, 276 (1955). Also New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. CX-20, 1955.